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Wavelet Transforms and Order-Two Densities of Fractals

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We highlight a correspondence between order-two densities and wavelet-like transforms of certain fractal measures. We use a variant of the ergodic theorem to demonstrate that these densities and transforms are well-behaved for a large class of quasi-self-similar fractals. We show that parallel ideas can be used to study the local behavior of certain fractal functions.

KEY WORDS: Wavelet transform; order-two averages; fractals; mixing repeller.

1. INTRODUCTION

The wavelet transform has been described as a "mathematical microscope" designed to study the local behavior of a function or a measure by integrating against translates and dilates of a given function called a *wavelet*.[1-3] Typically, a wavelet $w: \mathbb{R}^d \to \mathbb{R}$ is large close to the origin and small far away. The *wavelet transform* of a measure μ is defined to be

$$W(x, \varepsilon) = \int_{\mathbb{R}^d} \varepsilon^{-s} w\left(\frac{x-y}{\varepsilon}\right) d\mu(y) \qquad (x \in \mathbb{R}^d, \varepsilon > 0) \tag{1.1}$$

where s is a suitably chosen number. Thus when ε is small, $W(x, \varepsilon)$ reflects the nature of μ near x.

A number of authors have studied wavelet transforms of measures supported by fractals, and multifractal measures.[4-6] Choice of a suitable w depends on the purpose for which the wavelet transform is used. There is considerable divergence between authors as to the conditions that w ought to satisfy—for example, some require certain moments of w to vanish, while others specify rapid decrease at infinity.

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Here, we consider transforms by wavelets of an unconventional form, in that they are positive and have rather slow decay. The payoff is that in many cases the transforms (1.1) behave in a highly controlled manner as $\varepsilon \to 0$.

We study this behavior by relating wavelet transforms to the order-two densities introduced by Bedford and Fisher.[7] Their results on the existence of such densities in certain cases may then be interpreted in wavelet terms. We use a variation on the ergodic theorem to give a simplified proof of their results, which are extended to a more general setting.

In Section 5 we use similar techniques to examine the local behavior of certain fractal functions. Again, we may view this either in terms of order-two averages or in terms of wavelet-like transformations; ref. 8 for aspects of this latter approach.

2. DENSITIES AND TRANSFORMS OF MEASURES

Let μ be a finite Borel measure or "mass distribution" on $\mathbb{R}^d$ (or, more generally, on a d-dimensional Riemann manifold). For suitable s, we write

$$A(x, r) = \mu(B(x, r))/(2r)^s \tag{2.1}$$

where $B(x, r)$ is the closed ball of center x and radius r. Thus, $\lim_{r \to 0} A(x, r)$ is the density of μ at x. For "reasonably smoothly" distributed μ, this limit will exist for many x, but for more irregular mass distributions, it will not exist. For example, taking $s = \log 2/\log 3$ and μ as the natural "equidistributed" measure on the middle-third Cantor set E (thus, μ is obtained by repeated subdivision of a unit mass, or equivalently, is the restriction of the s-dimensional Hausdorff measure to E), then the upper limit is $2^{-\log 2/\log 3}$ but the lower limit is strictly less, μ-almost everywhere on E. It is natural to ask how $A(x, r)$ "oscillates" between its upper and lower limits as r tends to 0, and, in particular, what its "average" value is, in some sense. The Cantor set has obvious self-similarities at scales $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$, so it is natural to take an average that assigns equal weight to each of these scaling steps, i.e., an average on a logarithmic scale.

Thus, following Bedford and Fisher,[7] we define the *order-two averages* of μ at x as

$$A_2(x, T) = \frac{1}{T} \int_{t=0}^{T} A(x, e^{-t})\, dt \tag{2.2}$$

$$= 2^{-s} \frac{1}{T} \int_{t=0}^{T} \mu(B(x, e^{-t}))\, e^{st}\, dt \tag{2.3}$$

The *order-two density* of μ at x is defined to be $\lim_{T\to\infty} A_2(x, T)$. In Section 4 we shall see that this limit exists for μ-almost all x in a wide variety of cases, including the middle-third Cantor set measure mentioned above.

The following proposition relates the order-two averages of μ to a wavelet transform.

Proposition 2.1. Let μ be a finite measure on $\mathbb{R}^d$ such that

$$\mu(B(x, r)) \leqslant \alpha r^s \qquad (x \in \mathbb{R}^d, r > 0) \tag{2.4}$$

where $\alpha > 0$ and $s > 0$. Then

$$\int \frac{1}{\varepsilon^s |\log \varepsilon|} w\left(\frac{|x-y|}{\varepsilon}\right) d\mu(y) = 2^s s A_2(x, T) + o(1) \tag{2.5}$$

as $\varepsilon = e^{-T} \to 0$, where

$$w(r) = \begin{cases} 1, & 0 \leqslant r \leqslant 1 \\ r^{-s}, & 1 \leqslant r \end{cases} \tag{2.6}$$

Proof. On substituting $r = e^{-t}$ in (2.3), we get

$$\begin{aligned} 2^s A_2(x, T) &= \frac{1}{T} \int_{r=e^{-T}}^{1} r^{-s-1} \mu(B(x, r))\, dr \\ &= \frac{1}{|\log \varepsilon|} \int_{r=\varepsilon}^{1} r^{-s-1} m(r)\, dr \end{aligned}$$

where $\varepsilon = e^{-T}$ and $m(r) = \mu(B(x, r))$. Hence, integrating by parts,

$$|\log \varepsilon|\; 2^s A_2(x, T) = s^{-1}[m(\varepsilon)\, \varepsilon^{-s} - m(1)] + s^{-1} \int_{r=\varepsilon}^{1} r^{-s}\, dm(r)$$

so, using (2.4),

$$\begin{aligned} 2^s A_2(x, T) &= \frac{1}{|\log \varepsilon|} \int_{|x-y| \geqslant \varepsilon} \frac{d\mu(y)}{|x-y|^s} + o(1) \\ &= \int \frac{1}{\varepsilon^s |\log \varepsilon|} w\left(\frac{|x-y|}{\varepsilon}\right) d\mu(y) + o(1) \end{aligned} \tag{2.7}$$

where w is given by (2.6) and we have used (2.4) in neglecting the contribution of $\{y: |x-y| < \varepsilon\}$ to the integral in (2.7). ∎

In particular, it follows that

$$\lim_{T\to\infty} A_2(x, T) = 2^{-s}s^{-1} \lim_{\varepsilon\to 0} \int \frac{1}{\varepsilon^s\,|\log\varepsilon|}\, w\left(\frac{|x-y|}{\varepsilon}\right) d\mu(y) \qquad (2.8)$$

if this limit exists.

Notice that the wavelet (2.6) is "flattened" for $0 \leqslant r \leqslant 1$. This is compensated for by the introduction of the logarithmic term in the wavelet transform

$$W(x, \varepsilon) = \int \frac{1}{\varepsilon^s\,|\log\varepsilon|}\, w\left(\frac{|x-y|}{\varepsilon}\right) d\mu(y) \qquad (2.9)$$

This approach has several advantages, in particular, that in many situations of interest the integral (2.9) is bounded away from 0 and ∞ for small ε.

For one example of this, let E be an s-set, that is, a (Borel) subset of $\mathbb{R}^d$ with $0 < \mathscr{H}^s(E) < \infty$, where $\mathscr{H}^s$ denotes the s-dimensional Hausdorff measure (see ref. 9 for details of Hausdorff measures and dimension). Taking μ to be the restriction of $\mathscr{H}^s$ to E, we have that

$$A(x, r) = \frac{\mathscr{H}^s(E \cap B(x, r))}{(2r)^s}$$

Such densities have been studied for many years.[10–12] In particular, in the sense of $\mathscr{H}^s$-measure,

$$\lim_{r\to 0} A(x, r) = 0 \qquad (\text{a.a. } x \notin E)$$

$$\overline{\lim_{r\to 0}}\, A(x, r) \leqslant 1 \qquad (\text{a.a. } x \in E)$$

Moreover, if s is not an integer, or if s is an integer but E is "irregular," then there is a positive constant a depending only on s and d such that

$$\overline{\lim_{r\to 0}}\, A(x, r) - \underline{\lim_{r\to 0}}\, A(x, r) \geqslant a \qquad (\text{a.a. } x \in E)$$

Thus, in general, the density $\lim_{r\to 0} A(x, r)$ does not exist for $x \in E$.

On the other hand, for many sets E, including self-similar sets such as the middle-third Cantor set, it may be shown that there are positive numbers c_1, c_2 such that $c_1 \leqslant A(x, r) \leqslant c_2$ for all $x \in E$ and $r \leqslant 1$. Thus, $c_1 \leqslant A_2(x, T) \leqslant c_2$, so, by Proposition 2.1, $W(x, \varepsilon)$ is bounded away from 0 and ∞ for small ε. Moreover, as we shall see, $\lim_{\varepsilon\to 0} W(x, \varepsilon)$ exists for almost all x. Some other approaches to wavelet transforms of fractals use a wavelet w to define a number s_0 such that (1.1) tends to ∞ for $s < s_0$ and tends to 0 for $s > s_0$, without controlling the transform at the critical value s_0. The advantage of using (2.5)–(2.6) is that the transform is often of interest when $s = s_0$. We discuss such a situation in Section 4.

3. AN ERGODIC THEOREM

In this section we derive a variant of the Birkhoff ergodic theorem which is suited to several of the applications that we have in mind.

Proposition 3.1. Let S be a measure-preserving transformation on a finite measure space (X, v) and let $f_n \in L^1(X)$ $(n = 0, 1, 2,...)$. Let $\theta_n \in L^1(X)$ $(n = 0, 1, 2,...)$ and suppose that $\theta_n(x) \to 0$ for v-almost all x. Suppose that, for all sufficiently large n,

$$|f_n(S^k x) - f_{n+k}(x)| \leqslant \theta_n(x) \tag{3.1}$$

for all $x \in X$ and $k \in \mathbb{Z}^+$. Then $(1/n) \sum_{k=0}^{n-1} f_k(x)$ converges pointwise a.e. to a function $f \in L^1(X)$. Moreover, if S is ergodic, then f is a.e. constant.

Proof. For large enough n and $m \geqslant 1$, we have identically that

$$\frac{1}{m+n} \sum_{k=0}^{m+n-1} f_k(x) = \frac{1}{m+n} \sum_{k=0}^{n-1} f_k(x) \tag{3.2}$$

$$+ \frac{1}{m+n} \sum_{k=0}^{m-1} [f_{n+k}(x) - f_n(S^k x)] \tag{3.3}$$

$$+ \frac{m}{m+n} \frac{1}{m} \sum_{k=0}^{m-1} f_n(S^k x) \tag{3.4}$$

Letting $m \to \infty$ for fixed n, (3.2) converges to 0 a.e., (3.3) is bounded in modulus by $\theta_n(x)$ for each x, and (3.4) converges a.e. to a function $f_n^* \in L^1(X)$, by the Birkhoff ergodic theorem.[13] Hence

$$f_n^*(x) - \theta_n(x) \leqslant \varliminf_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} f_k(x)$$

$$\leqslant \varlimsup_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} f_k(x) \leqslant f_n^*(x) + \theta_n(x)$$

for a.a. x. Letting $n \to \infty$, it follows that

$$\lim_{n \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} f_k(x) = \lim_{n \to \infty} f_n^*(x) \equiv f(x)$$

say, for a.a. x, where

$$|f(x)-f_n^*(x)|\leqslant\theta_n(x)\quad \text{a.e.},$$

so $f\in L^1(X)$

If S is ergodic, then f_n^* is constant a.e. for each n, so f is constant a.e. ∎

4. MIXING REPELLERS

The aim of this section is to demonstrate the existence of the order-two densities, equivalently the limit of (2.9), as $\varepsilon\to 0$, of a class of mixing repellers that includes many hyperbolic Julia sets. (We take μ to be the appropriate dimensional Hausdorff measure on the sets.)

Proposition 4.1 was obtained in the special case of a one-dimensional two-part cookie-cutter by Bedford and Fisher.(7) Our proof avoids some of the technicalities of theirs by estimating integrals of $A(x, r)$ rather than $A(x, r)$ directly.

Many standard properties of mixing repellers may be deduced from the existence of Markov partitions, using symbolic dynamics.(14,15) Our approach has the advantage that we can take such properties as our starting point, rather than having to work from the Markov partitions themselves.

Let M be a d-dimensional Riemann manifold and S a conformal map of class $C^{1+\eta}$ on M, i.e., with the tangent map of S satisfying a Hölder condition of exponent $\eta>0$. Let J be a *mixing repeller* for S, i.e., J is a subset of M satisfying the following conditions.

(i) S is expanding, i.e., there exist $c>0$ and $\alpha>1$ such that

$$|DS^n(x)|\geqslant c\alpha^n \tag{4.1}$$

for all $x\in J$ and $n\geqslant 1$, where $DS^n(x)$ is the tangent map of S^n at x. [Since S is conformal, $DS^n(x)$ is a similarity transformation, with $|DS^n(x)|$ the similarity ratio.]

(ii) There is an open neighbourhood V of J such that

$$J=\{x\in V: S^n x\in V \text{ for all } n\geqslant 0\}$$

(iii) S is topologically mixing on J, i.e., if U is an open set that intersects J, then $J\subseteq f^n(U)$ for some $n>0$.

These conditions imply that J is invariant under S, i.e., $S^{-1}(J)=S(J)=J$. The best-known examples of mixing repellers are hyperbolic Julia sets of suitable meromorphic mappings S on $\mathbb{C}$.

A fundamental property of such a mixing repeller J is that $0 < \mathscr{H}^s(J) < \infty$, where s is the Hausdorff dimension of J and $\mathscr{H}^s$ is the s-dimensional Hausdorff measure; see refs. 14 and 16 for different proofs of this.

Our aim is to show that the order-two densities of the restriction of $\mathscr{H}^s$ to J exist and are constant $\mathscr{H}^s$-almost everywhere on J.

We list some facts about mixing repellers that are by now regarded as standard. These may be gleaned from various of refs. 7 and 14–18.

1. There exists $r_0 > 0$, $c' > 0$, and $\alpha_1 > 1$ such that for all $x \in J$, $0 < r \leqslant r_0$, and $n \in \mathbb{Z}^+$ we have

$$S^{-n}(B(x, r)) \subseteq V$$

and

$$S^n\colon\ S^{-n}(B(S^n x, r))\,|_x \to B(S^n x, r) \tag{4.2}$$

is a $C^{1+\eta}$ bijection satisfying

$$|DS^n(y)| \geqslant c'\alpha_1^n \tag{4.3}$$

on $S^{-n}(B(S^n x, r))\,|_x$, the connected component of $S^{-n}(B(S^n x, r))$ containing x.

2. The following form of the "bounded distortion property" holds. There exist $r_0 > 0$ and $a > 0$ such that

$$|\log |DS^k(y_1)| - \log |DS^k(y_2)|| \leqslant a\, |S^k y_1 - S^k y_2|^\eta \tag{4.4}$$

for all $y_1,\ y_2 \in S^{-k}(B(S^k x, r))\,|_x$, for all $k \in \mathbb{Z}^+$, $x \in J$, and $r \leqslant r_0$.

3. There exists $r_0 > 0$ such that for all $n \in \mathbb{Z}^+$, $x \in J$, and $r \leqslant r_0$

$$S^n B(x, r/|DS^n(x_+)|) \subseteq B(S^n x, r) \subseteq S^n B(x, r/|DS^n(x_-)|) \tag{4.5}$$

where $x_-,\ x_+ \in S^{-n}(B(S^n x, r))\,|_x$ are chosen so that

$$|DS^n(x_-)| = \inf_y |DS^n(y)|, \qquad |DS^n(x_+)| = \sup_y |DS^n(y)| \tag{4.6}$$

with the inf and sup taken as y varies over $S^{-n}(B(S^n x, r))\,|_x$. This follows by applying the mean value theorem to the restrictions of S^n and S^{-n} to the domains in (4.2).

4. We have the conformal mapping property: if ϕ is a mapping that is a C^1 conformal bijection on $\bar{U}$, where U is an open domain, then

$$\mathscr{H}^s(\phi(E)) = \int_{y \in E} |D\phi(y)|^s\, d\mathscr{H}^s(y)$$

where $\mathscr{H}^s$ is the s-dimensional Hausdorff measure and E is an $\mathscr{H}^s$-measurable subset of $\bar{U}$. In particular, we have that if $E \subset \bar{U}$,

$$\mathscr{H}^s(E) \inf_{y \in U} |D\phi(y)|^s \leqslant \mathscr{H}^s(\phi(E)) \leqslant \mathscr{H}^s(E) \sup_{y \in U} |D\phi(y)|^s \tag{4.7}$$

5. There are positive constants c_1, c_2 such that

$$c_1 \leqslant r^{-s} \mathscr{H}^s(B(x, r) \cap J) \leqslant c_2 \tag{4.8}$$

for all $x \in J$ and $0 < r \leqslant 1$, where s is the Hausdorff dimension of J.

Proposition 4.1. Let J be a mixing repeller of a $C^{1+\eta}$ conformal map S on a Riemann manifold M, as above, and let s be the Hausdorff dimension of J. Let ν be an invariant measure on J. Then there exists $c > 0$ such that the order-two density $\lim_{T \to \infty} A_2(x, T)$ exists and equals c for ν-almost all $x \in J$.

Proof. Let $r_0 > 0$ be chosen small enough to ensure that conditions 1–5 above are satisfied. For $n = 0, 1, 2,\dots$ and $x \in J$, let

$$f_n(x) = \int_{\log |DS^n(x)|}^{\log |DS^{n+1}(x)|} e^{st} \mu(B(x, e^{-t}))\, dt$$

where μ is the restriction of $\mathscr{H}^s$ to J.

Providing that n is large enough to ensure that $|DS^n(x)| \geqslant r_0^{-1}$ for all $x \in J$, we have, with

$$x_-, x_+ \in S^{-k}(B(S^k x, |DS^n(S^k x)|^{-1}))\,|_x \subseteq S^{-k}(B(S^k x, c^{-1}\alpha^{-n}))\,|_x \tag{4.9}$$

[using (4.1)], that

$$\begin{aligned} f_n(S^k x) &= \int_{\log |DS^n(S^k x)|}^{\log |DS^{n+1}(S^k x)|} e^{st} \mu(B(S^k x, e^{-t}))\, dt \\ &\leqslant \int_{\log |DS^n(S^k x)|}^{\log |DS^{n+1}(S^k x)|} e^{st} \mu(S^{-k} B(S^k x, e^{-t}))\,|_x\, |DS^k(x_+)|^s\, dt \\ &\leqslant \int_{\log |DS^n(S^k x)|}^{\log |DS^{n+1}(S^k x)|} e^{st} \mu(B(x, e^{-t}/|DS^k(x_-)|))\, |DS^k(x_+)|^s\, dt \end{aligned}$$

using (4.7) and (4.5). Substituting $u = t + \log |DS^k(x_-)|$, this becomes

$$f_n(S^k x) \leqslant \int_{\log |DS^n(S^k x)| + \log |DS^k(x_-)|}^{\log |DS^{n+1}(S^k x)| + \log |DS^k(x_-)|} e^{su} \mu(B(x, e^{-u})) \left| \frac{DS^k(x_+)}{DS^k(x_-)} \right|^s du$$

Noting (4.8) and (4.9) and using (4.4) both to change the range of integration and to estimate the integrand, we obtain

$$f_n(S^k x) \leqslant \left[\int_{\log |DS^n(S^k x)| + \log |DS^k(x)|}^{\log |DS^{n+1}(S^k x)| + \log |DS^k(x)|} e^{su} \mu(B(x, e^{-u}))\, du + 2c_2 a c^{-\eta} \alpha^{-n\eta} \right] \times \exp[as(2c^{-1}\alpha^{-n})^{\eta}]$$

Since

$$\log |DS^n(S^k x)| + \log |DS^k(x)| = \log |DS^{n+k}(x)|$$

we get, using (4.8) again, that

$$f_n(S^k x) \leqslant f_{n+k}(x) + \varepsilon_n$$

where $\varepsilon_n \to 0$ as $n \to \infty$. This is half of the inequality

$$|f_n(S^k x) - f_{n+k}(x)| \leqslant \varepsilon_n \qquad (x \in J)$$

for all sufficiently large n; the other half follows in exactly the same way.

Thus

$$\frac{1}{n} \int_0^{\log |DS^n(x)|} e^{st} \mu(B(e^{-t}, x))\, dt = \frac{1}{n} \sum_{k=0}^{n-1} f_k(x) \to c_0$$

pointwise ν-almost everywhere, for some c_0, using Proposition 3.1 and noting that a mixing invariant measure is ergodic. By (4.8), $0 < c_1 \leqslant c_0 \leqslant c_2 < \infty$.

By the simplest case from the theory of Lyapunov exponents (which follows from a simple application of the ergodic theorem), we have that, ν-almost everywhere, $(1/n) \log |DS^n(x)| \to \gamma$ for some $\gamma > 0$; hence

$$\frac{1}{\log |DS^n(x)|} \int_0^{\log |DS^n(x)|} e^{st} \mu(B(e^{-t}, x))\, dt \to \frac{c_0}{\gamma}$$

as $n \to \infty$. The result follows using (4.8) to pass from the discrete to the continuous limit. ▮

Corollary 4.2. Let J be a mixing repeller of a conformal mapping S, as above. Then there exists $c > 0$ such that the order-two density $\lim_{T \to \infty} A_2(x, T)$ exists and equals c for $\mathscr{H}^s$-almost all $x \in J$.

Proof. We note that there is an ergodic invariant measure on J, namely the Gibbs measure, that is equivalent to $\mathscr{H}^s$.[14] The corollary follows immediately. ▮

In view of (2.5), Proposition 4.1 implies that in this situation $\lim_{\varepsilon \to 0} W(x, \varepsilon)$ exists and is constant a.e., where W is given by (2.6) and (2.9).

Explicit values of order-two densities seem hard to calculate. Patzschke and Zähle[19] have obtained a value of 0.48272... in the case of the middle-third Cantor set. It would be of interest to know whether the order-two densities have an interpretation in terms of the thermodynamic formalism.

5. SOME FRACTAL FUNCTIONS

In this section we show how the preceding analysis may be adapted to study the local oscillation of "fractal functions" such as

$$g(x) = \sum_{m=0}^{\infty} 2^{-am} \sin 2^m x \qquad (x \in \mathbb{R}) \tag{5.1}$$

where a $(0 < a < 1)$ is a fixed parameter. Such functions were introduced by Weierstrass in his construction of a continuous but nowhere differentiable function.[8,9,20] It may be shown that g satisfies a Hölder condition

$$|g(x+u) - g(x)| \leqslant cu^a$$

and, further, that the box-counting dimension of the graph of g is $2 - a$ (see ref. 9, Example 11.3). (It is conjectured, but not yet proved, that the Hausdorff dimension of the graph is also $2 - a$.)

We introduce the average local moments of g. For a fixed $p \geqslant 1$, let

$$A(x, r) = r^{-1-ap} \int_{u=0}^{r} |g(x+u) - g(x)|^p \, du \tag{5.2}$$

The argument of ref. 9, Example 11.3, shows that there are numbers c_1, c_2 such that

$$0 < c_1 \leqslant A(x, r) \leqslant c_2 < \infty \tag{5.3}$$

for all $x \in \mathbb{R}$ and $0 < r \leqslant 1$. However, for fixed x, the averages $A(x, r)$ behave in an oscillatory manner for small r. To obtain convergence to a limit, we need to use order-two averages

$$A_2(x, T) = \frac{1}{T} \int_{t=0}^{T} A(x, e^{-t}) \, dt \tag{5.4}$$

By (5.3)

$$0 < c_1 \leqslant A_2(x, T) \leqslant c_2 < \infty \tag{5.5}$$

We may express these order-two averages in a wavelet-like form by a very similar method to that employed in Proposition 2.1. We get that

$$A_2(x, T) = \frac{1}{1+ap} \int \frac{1}{\varepsilon^{1+ap} |\log \varepsilon|} w\left(\frac{y-x}{\varepsilon}\right) |f(y) - f(x)|^p \, dy + o(1) \tag{5.6}$$

where $\varepsilon = e^{-T}$ and

$$w(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & 0 < x \leqslant 1 \\ x^{-(1+ap)}, & 1 \leqslant x \end{cases} \tag{5.7}$$

By virtue of (5.5), this is, in a sense, the "correct" wavelet function to use.

Proposition 5.1. Let g be given by (5.1) and let $p \geqslant 1$. Then there exists $c_0 > 0$ such that $\lim_{T \to \infty} A_2(x, T)$ exists and equals c_0 for a.a. $x \in \mathbb{R}$ (in the sense of Lebesgue measure), where $A_2(x, T)$ is the order-two average moment defined by (5.2) and (5.4).

Proof. Define S: $[0, 2\pi) \to [0, 2\pi)$ by

$$Sx = 2x \pmod{2\pi} \tag{5.8}$$

From (5.1)

$$g(Sx) = 2^a g(x) - 2^a \sin x, \qquad x \in [0, 2\pi) \tag{5.9}$$

Hence, using (5.2) and substituting $u' = \frac{1}{2}u$,

$$A(Sx, r) = (\tfrac{1}{2}r)^{-1-ap} \int_{u=0}^{r/2} |g(x+u) - g(x) - \sin(x+u) + \sin x|^p \, du$$

Applying Hölder's inequality and comparing with (5.2) gives

$$|A(Sx, r) - A(x, \tfrac{1}{2}r)| \leqslant c r^{p(1-a)} \tag{5.10}$$

where c is indepenent of x and r.

Let

$$f_n(x) = \int_{t=n}^{n+1} A(x, 2^{-t}) \, dt$$

Then

$$|f_n(Sx)-f_{n+1}(x)| = \left| \int_{t=n}^{n+1} A(Sx, 2^{-t})\,dt - \int_{t=n}^{n+1} A(x, \tfrac{1}{2}2^{-t})\,dt \right|$$
$$\leqslant c2^{-np(1-a)}$$

by (5.10). Iterating, we obtain

$$|f_n(S^k x) - f_{n+k}(x)| \leqslant c' 2^{-np(1-a)}$$

where c' is independent of n, k, and x. Since S is measure-preserving and ergodic with respect to the Lebesgue measure on $[0, 2\pi)$, Proposition 3.1 gives that

$$\frac{1}{n}\int_{t=0}^{n} A(x, 2^{-t})\,dt = \frac{1}{n}\sum_{k=0}^{n-1} f_k(x) \to c_0$$

pointwise a.e. for some number c_0. Using (5.3), we can pass to the continuous limit, and a change of variable gives the result, noting (5.5). ∎

Obviously, this leads to the existence and constancy a.e. of the limit as $\varepsilon \to 0$ of the "wavelet transform" on the right-hand side of (5.6).

Precisely the same argument may be employed replacing (5.2) by, for example,

$$A(x, r) = r^{-1-a} \int_{u=0}^{r} [g(x+u) - g(x)]\,du$$

Again, $\lim_{T\to\infty} A_2(x, T)$ exists and is constant a.e., but in this case the linearity in $g(x+u) - g(x)$ ensures that this constant equals 0.

It is also possible to study "local Fourier transforms," for example, with

$$A(x, r) = r^{-1-a} \int_{u=0}^{r} [g(x+u) - g(x)]\, e^{i\lambda u}\,du$$

Apart from certain exceptional values of λ, we again have that $\lim_{T\to\infty} A_2(x, T) = 0$ for almost all x.

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